CHAPTER 2

Electrical components and circuits

Chapter objectives

When you have finished this chapter you should be able to:

- understand the basic electrical components: resistor, capacitor, and inductor;
- deal with resistive elements using the node voltage method and the node voltage analysis method;
- deal with resistive elements using the mesh current method, principle of superposition, as well as Thévenin and Norton equivalent circuits;
- deal with sinusoidal sources and complex impedances.

2.1 Introduction

Most mechatronic systems contain electrical components and circuits, hence a knowledge of the concepts of electric charge (Q), electric field (E), and magnetic field (B), as well as, potential (V) is important. We will not be concerned with a detailed description of these quantities but will use approximation methods when dealing with them. Electronics can be considered as a more practical approach to these subjects.

The fundamental quantity in electronics is electric charge, which, at a basic level, is due to the charge properties of the fundamental particles of matter. For all intents and purposes it is the electrons (or lack of electrons) that matter. The role of the proton charge is negligible.

The aggregate motion of charge, the current (I), is given as

$$I(t) = \frac{dQ}{dt}, \tag{2.1}$$

where dQ is the amount of *positive* charge crossing a specified surface in a time dt. It is accepted that the charges in motion are actually negative electrons. Thus the electrons move in the opposite direction to the current flow. The SI unit for current is the ampere (A). For most electronic circuits the ampere is a rather large unit so the milliampere (mA), or even the microampere (μA), unit is more common.

Current flowing in a conductor is due to a potential difference between its ends. Electrons move from a point of less positive potential to more positive potential and the current flows in the opposite direction.

It is often more convenient to consider the electrostatic potential (V) rather than the electric field (E) as the motivating influence for the flow of electric charge. The generalized vector properties of E are usually not important. The change in potential dV across a distance dx in an electric field is

$$dV = -E \times dx. \tag{2.2}$$

A positive charge will move from a higher to a lower potential. The potential is also referred to as the potential difference, or (incorrectly) as just voltage:

$$V = V_{21} = V_2 - V_1 = \int_{V_1}^{V_2} dV. \tag{2.3}$$

The SI unit of potential difference is the volt (V). Direct current (d.c.) circuit analysis deals with constant currents and voltages, while alternating current (a.c.) circuit analysis deals with time-varying voltage and current signals whose time average values are zero.

Circuits with time-average values of non-zero are also important and will be mentioned briefly in the section on filters. The d.c. circuit components considered in this book are the constant voltage source, constant current source, and the resistor.

Figure 2.1 is a schematic diagram consisting of *idealized* circuit elements encountered in d.c. circuits, each of which represents some property of the *actual* circuit.

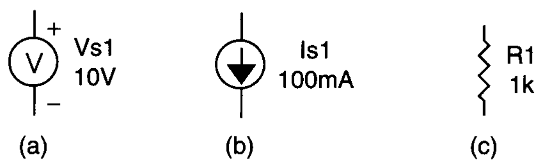

Figure 2.1 Common elements found in d.c. circuits: (a) ideal voltage source; (b) ideal current source; (c) resistor.

2.1.1 External energy sources

Charge can flow in a material under the influence of an external electric field. Eventually the internal field due to the repositioned charge cancels the external electric field resulting in zero current flow. To maintain a potential drop (and flow of charge) requires an electromagnetic force (EMF), that is, an external energy source (battery, power supply, signal generator, etc.).

There are basically two types of EMFs that are of interest:

- the *ideal voltage source*, which is able to maintain a constant voltage regardless of the current it must put out ($I \to \infty$ is possible);
- the *ideal current source*, which is able to maintain a constant current regardless of the voltage needed ($V \to \infty$ is possible).

Because a battery cannot produce an infinite amount of current, a suitable model for the behavior of a battery is an internal resistance in series with an ideal voltage source (zero resistance). Real-life EMFs can always be approximated with ideal EMFs and appropriate combinations of other circuit elements.

2.1.2 Ground

A voltage must always be measured relative to some reference point. We should always refer to a voltage (or potential difference) being 'across' something, and simply referring to voltage at a point assumes that the voltage point is stated with respect to ground. Similarly current flows through something, by convention, from a higher potential to a lower (do not refer to the current 'in' something). Under a strict definition, ground is the body of the Earth (it is sometimes referred to as earth). It is an infinite electrical sink. It can accept or supply any reasonable amount of charge without changing its electrical characteristics.

It is common, but not always necessary, to connect some part of the circuit to earth or ground, which is taken, for convenience and by convention, to be at zero volts. Frequently, a common (or reference) connection from, and electrical current to, the metal chassis of a piece of equipment suffices. Sometimes there is a *common* reference voltage that is not at 0 V. Figure 2.2 show some common ways of depicting ground on a circuit diagram.

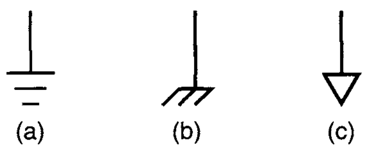

Figure 2.2 Some grounding circuit diagram symbols: (a) earth ground; (b) chassis ground; (c) common.

When neither a ground nor any other voltage reference is shown explicitly on a schematic diagram, it is useful for purposes of discussion to adopt the convention that the bottom line on a circuit is at zero potential.

2.2 Electrical components

The basic electrical components which are commonly used in mechatronic systems include resistors, capacitors, and inductors. The properties of these elements are now discussed.

2.2.1 Resistance

Resistance is a function of the material and shape of the object, and has SI units of ohms (Ω). It is more common to find units of kilohm (kΩ) and megohm (MΩ). The inverse of resistivity is conductivity.

Resistor tolerances can be as much as ±20 percent for general-purpose resistors to ±0.1 percent for ultra-precision resistors. Only wire-wound resistors are capable of ultra-precision accuracy.

For most materials:

$$V \propto I; \quad V = RI, \tag{2.4}$$

where $V = V_2 - V_1$ is the voltage *across* the object, I is the current *through* the object, and R is a proportionality constant called the resistance of the object. This is Ohm's law.

The resistance in a uniform section of material (for example, a wire) depends on its length L, cross-sectional area A, and the resistivity of the material ρ, so that

$$R = \rho \frac{L}{A}, \tag{2.5}$$

where the resistivity has units of ohm-m (Ω-m). Restivitiy is the basic property that defines a material's capability to resist current flow. Values of resistivity for selected materials are given in Table 2.1.

It is more convenient to consider a material as conducting electrical current rather than resisting its flow. The conductivity of a material, σ, is simply the reciprocal of resistivity:

$$\text{Electrical conductivity, } \sigma = \frac{1}{\rho}. \tag{2.6}$$

Table 2.1 Resistivity of selected materials

Material	*Resistivity (Ω-m)*
Conductors	10^{-8}
Aluminum	2.8
Aluminum alloys	4.0
Cast iron	65.0
Copper	1.7
Gold	2.4
Iron	9.5
Lead	20.6
Magnesium	4.5
Nickel	6.8
Silver	1.6
Steel, low C	17.0
Steel, stainless	70.0
Tin	11.5
Zinc	6.0
Carbon	5000
Semiconductors	10^1 to 10^5
Silicon	10×10^3
Insulators	10^{12} to 101^5
Natural rubber	1.0×10^{12}
Polyethylene	100×10^{12}

Conductivity has units of $(\Omega\text{-m})^{-1}$.

Table 2.2 shows the resistor color code. Using this table, it is easy to determine the resistance value and tolerance of a resistor that is color-coded (Figure 2.3).

Table 2.2 Resistor color code

Color	*Value*	*Color*	*Value*
Black	0	Gold	±5%
Brown	1	Silver	±10%
Red	2	nothing	±20%
Orange	3		
Yellow	4		
Green	5		
Blue	6		
Violet	7		
Gray	8		
White	9		

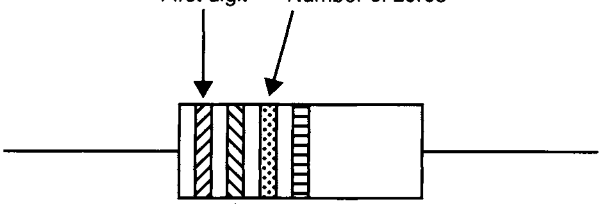

Figure 2.3 Resistor color code.

EXAMPLE 2.1 Resistance

Determine the resistance of a silver wire, which is 0.5 m long and 1.5 mm in diameter.

Solution

$$R = \rho \frac{L}{A} = 1.6 \times 10^{-8} \frac{0.500}{\pi \frac{(0.0015)^2}{4}} = 0.00453 = 4.5\ \text{m}\Omega \tag{2.6A}$$

EXAMPLE 2.2 Resistance color code

Determine the possible range of resistance values for the following color band: orange, gray, and yellow.

Solution

From Table 2.2, orange color has a value of 3, gray color has a value of 8, and yellow color has a value of 4. Hence, the resistance is 38×10^4 (380 kΩ), with tolerance of $\pm 20\% \times 380$, or (380 ± 76) kΩ, so that $304\ \text{k}\Omega \leq R \leq 456\ \text{k}\Omega$.

2.2.2 Capacitance

The fundamental property of a capacitor is that it can store charge and, hence, electric field energy. The capacitance C between two appropriate surfaces is found from

$$V = \frac{Q}{C}, \tag{2.7}$$

where V is the potential difference between the surfaces and Q is the magnitude of the charge distributed on either surface. In terms of current, $I = dQ/dt$ implies

$$\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt} \tag{2.8}$$

In electronics, we take $I = I_D$ (displacement current). In other words, the current flowing from or to the capacitor is taken to be equal to the displacement current through the capacitor. Consequently, capacitors add linearly when placed in parallel.

There are four principal functions of a capacitor in a circuit:

- since Q can be stored, a capacitor can be used as a (non-ideal) source of I;
- since E can be stored a capacitor can be used as a (non-ideal) source of V;
- since a capacitor passes alternating current (a.c.) but not direct current (d.c.) it can be used to connect parts of a circuit that must operate at different d.c. voltage levels;
- a capacitor and resistor in series will limit current and hence smooth sharp edges in voltage signals.

Charging or discharging a capacitor with a constant current results in the capacitor having a voltage signal with a constant slope, i.e.

$$\frac{dV}{dt} = \frac{I}{C} = \text{constant}, \tag{2.8A}$$

if I is a constant.

Some capacitors (electrolytic) are asymmetric devices with a polarity that must be taken into account when placed in a circuit. The SI unit for capacitance is the farad (F). The capacitance in a circuit is typically measured in microfarads (μF) or picofarads (pF). Non-ideal circuits will have stray capacitance, leakage currents and inductive coupling at high frequency. Although important in real circuit design, we will not go into greater detail of these aspects at this point.

Capacitors can be obtained in various tolerance ratings from ±20 percent to ±0.5 percent. Because of dimensional changes, capacitors are highly temperature dependent. A capacitor does not hold a charge indefinitely because the dielectric is never a perfect insulator. Capacitors are rated for leakage, the conduction through the dielectric, by the leakage resistance–capacitance product (MΩ–μF). High temperature increases leakage.

2.2.3 Inductance

Faraday's laws of electromagnetic induction applied to an inductor states that a changing current induces a back EMF that opposes the change. Putting this in another way,

$$V = V_A - V_B = L\frac{dI}{dt}, \tag{2.9}$$

where V is the voltage across the inductor and L is the inductance measured in henries (H). The more common units encountered in circuits are the microhenry (μH) and the millihenry (mH). The inductance will tend to smoothen sudden changes in current just as the capacitance smoothens sudden changes in voltage. Of course, if the current is constant there will be no induced EMF. Hence, unlike the capacitor which behaves like an open-circuit in d.c. circuits, an inductor behaves like a short-circuit in d.c. circuits.

Applications using inductors are less common than those using capacitors, but inductors are very common in high frequency circuits. Inductors are never pure (ideal) inductances because they always have some resistance in and some capacitance between the coil windings. We will skip the effect these have on a circuit at this stage.

When choosing an inductor (occasionally called a choke) for a specific application, it is necessary to consider the value of the inductance, the d.c. resistance of the coil, the current-carrying capacity of the coil windings, the breakdown voltage between the coil and the frame, and the frequency range in which the coil is designed to operate. To obtain a very high inductance it is necessary to have a coil of many turns. Winding the coil on a closed-loop iron or ferrite core further increases the inductance. To obtain as pure an inductance as possible, the d.c. resistance of the windings should be reduced to a minimum. Increasing the wire size, which, of course, increases the size of the choke, is the means of achieving this. The size of the wire also determines the current-handling capacity of the choke since the work done in forcing a current through a resistance is converted to heat in the resistance. Magnetic losses in an iron core also account for some heating, and this heating restricts any choke to a certain safe operating current. The windings of the coil must be insulated from the frame as well as from each other. Heavier insulation, which necessarily makes the choke more bulky, is used in applications where there will be a high voltage between the frame and the winding. The losses sustained in the iron core increases as the frequency increases. Large inductors, rated in henries, are used principally in power applications. The frequency in these circuits is relatively low, generally 60 Hz or low multiples thereof. In high-frequency circuits, such as those found in FM radios and television sets, very small inductors (of the order of microhenries) are often used.

Now that we have briefly familiarized ourselves with these basic electrical elements, it is now necessary to consider the basic techniques for analyzing them.

2.3 Resistive circuits

The basic techniques for the analysis of resistive circuits are:

- node voltage and mesh current analysis;
- the principle of superposition;
- Thévenin and Norton equivalent circuits.

The principle of superposition is a conceptual aid that can be very useful in visualizing the behavior of a circuit containing multiple sources. Thévenin and Norton equivalent circuits are the reductions of an arbitrary circuit to an equivalent, simpler circuit. In this section it will be shown that it is generally possible to reduce all linear circuits to one of two equivalent forms, and that any linear circuit analysis problem can be reduced to a simple voltage or current divider problem.

2.3.1 Node voltage method

Node voltage analysis is the most general method for the analysis of electrical circuits. In this section its application to linear resistive circuits will be illustrated. The node voltage method is based on defining the voltage at each node as an independent variable. One of the nodes is selected as a reference node (usually, but not necessarily, ground), and each of the other node voltages is referenced to this node. Once each node voltage is defined, Ohm's law may be applied between any two adjacent nodes in order to determine the current flowing in each branch. In the node voltage method, each branch current is expressed in terms of one or more node voltages; thus, currents do not explicitly enter into the equations. Figure 2.4(a) illustrates how one defines branch currents in this method.

In the node voltage method, we define the voltages at nodes *a* and *b* as v_a and v_b, respectively; the branch current flowing from *a* to *b* is then expressed in terms of these node voltages.

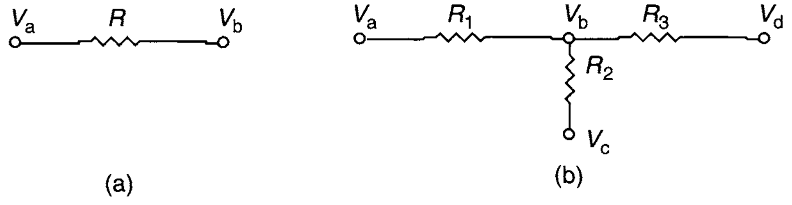

Figure 2.4 Use of Kirchhoff's current law in nodal analysis.

Once each branch current is defined in terms of the node voltages, Kirchhoff's current law (KCL) is applied at each node, so $\Sigma i = 0$.

Figure 2.4(b) illustrates this procedure for a more complex network. By KCL: $i_1 - i_2 - i_3 = 0$, where i_n is the current flowing through R_n. In the node voltage method, we express KCL by

$$\frac{v_a - v_b}{R_1} = \frac{v_b - v_c}{R_2} + \frac{v_b - v_d}{R_3} \tag{2.10}$$

Applying this method systematically to a circuit with n nodes would lead to obtaining n linear equations. However, one of the node voltages is the reference voltage and is therefore already known, since it is usually assumed to be zero. Thus, we can write $n - 1$ independent linear equations in the $n - 1$ independent variables (which, in this case, are the node voltages). Nodal analysis provides the minimum number of equations needed to solve the circuit, since any branch voltage or current may be determined from a knowledge of nodal voltages.

2.3.1.1 *Node voltage analysis method*

The steps involved in the node voltage analysis method are as follows:

1. Select a reference node (usually ground). Reference all other node voltages to this node.
2. Define the remaining $n - 1$ node voltages as the independent variables.
3. Apply KCL at each of the $n - 1$ nodes, expressing each current in terms of the adjacent node voltages.
4. Solve the linear system of $n - 1$ equations in $n - 1$ unknowns.

Let us now apply this method to a problem to illustrate the technique.

EXAMPLE 2.3 Node voltage analysis

In the circuit shown in Figure 2.5, $R_1 = 1\,\mathrm{k\Omega}$, $R_2 = 2\,\mathrm{k\Omega}$, $R_3 = 5\,\mathrm{k\Omega}$, and $i_S = 50\,\mathrm{mA}$. Determine the two node voltages.

Solution

The direction of current flow is selected arbitrarily (we assume that i_S is a positive current). We apply KCL at node a, to yield:

$$i_s - i_1 - i_2 = 0 \tag{2.11}$$

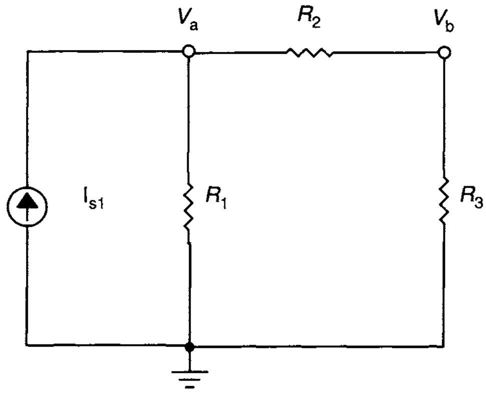

Figure 2.5 Example of nodal analysis.

Whereas, at node *b*,

$$i_2 - i_3 = 0 \tag{2.12}$$

There is no need to apply KCL at the reference node since the equation obtained at node *c*,

$$i_2 - i_3 = 0 \tag{2.13}$$

is not independent of Equations 2.11 and 2.12.

In a circuit containing n nodes, we can write at most $n-1$ independent equations.

When we apply the node voltage method, the currents i_1, i_2, and i_3 are expressed as functions of v_a, v_b, and v_c, the independent variables. Applying Ohm's law gives the following results:

$$i_1 = \frac{v_a - v_c}{R_1}, \tag{2.14}$$

since it is the potential difference, $v_a - v_c$, across R_1 that causes the current i_1 to flow from node *a* to node *c*. In the same manner,

$$\begin{aligned} i_2 &= \frac{v_a - v_b}{R_2} \\ i_3 &= \frac{v_b - v_c}{R_3}. \end{aligned} \tag{2.15}$$

Substituting the expression for the three currents in the nodal equations (equations 2.11 and 2.12, and noting that $v_c = 0$), leads to the following relationships:

$$i_s - \frac{v_a}{R_1} - \frac{v_a - v_b}{R_2} = 0 \tag{2.16}$$

and

$$\frac{v_a - v_b}{R_2} - \frac{v_b}{R_2} = 0. \tag{2.17}$$

We now solve these equations for v_a and v_b, for the given values of i, R_1, R_2, and R_3. The same equations are expressed as follows:

$$\begin{aligned} \left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_a + \left(-\frac{1}{R_2}\right) v_b &= i_s \\ \left(-\frac{1}{R_2}\right) v_a + \left(\frac{1}{R_2} + \frac{1}{R_3}\right) v_b &= 0. \end{aligned} \tag{2.18}$$

On substituting the given values,

$$\begin{aligned} \left[\left(\frac{1}{1} + \frac{1}{2}\right) v_a + \left(-\frac{1}{2}\right) v_b\right] \times 10^{-3} &= 50 \times 10^{-3} \\ \left[\left(-\frac{1}{2}\right) v_a + \left(\frac{1}{1} + \frac{1}{2}\right) v_b\right] \times 10^{-3} &= 0, \end{aligned} \tag{2.18A}$$

yielding two simultaneous equations:

$$1.5 v_a - 0.5 v_b = 50$$

and

$$-0.5 v_a - 0.7 v_b = 0$$

Solving these two equations leads to the following node voltages: $v_a = 43.75\,\mathrm{V}$ and $v_b = 31.25\,\mathrm{V}$.

2.3.2 Mesh current method

The second method of circuit analysis that we discuss employs the mesh currents as the independent variables; it is in many respects analogous to the method of node voltages. In this method, we write the appropriate number of independent equations, using mesh currents as the independent variables. Analysis by mesh currents consists of defining the currents around the individual meshes as the

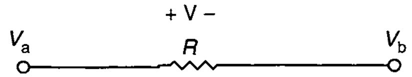

Figure 2.6 Basic principle of mesh analysis.

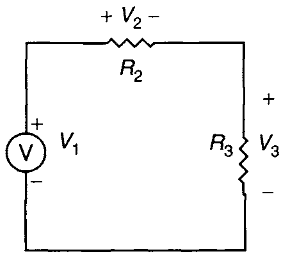

Figure 2.7 Use of Kirchoff's voltage law in mesh analysis.

independent variables. Then, the Kirchhoff's voltage law (KVL) is applied around each mesh to provide the desired system of equations.

In the mesh current method, we observe that a current flowing through a resistor in a specified direction defines the polarity of the voltage across the resistor, as illustrated in Figure 2.6, and that the sum of the voltages around a closed circuit must equal zero, by KVL. The current *i*, defined as flowing from left to right in Figure 2.6 establishes the polarity of the voltage across *R*. Once a convention is established regarding the direction of current flow around a mesh, simple application of KVL provides the desired equation. Figure 2.7 illustrates this point.

The number of equations obtained by this technique is equal to the number of meshes in the circuit. All branch currents and voltages may subsequently be obtained from the mesh currents. Since meshes are easily identified in a circuit, this method provides a very efficient and systematic procedure for the analysis of electrical circuits.

Once the direction of current flow has been selected, KVL requires that $v_1 - v_2 - v_3 = 0$.

2.3.2.1 *Mesh current analysis method*

The mesh current analysis method is described in the following steps:

1. Define each mesh current consistently. We shall always define mesh currents clockwise, for convenience.

2. Apply KVL around each mesh, expressing each voltage in terms of one or more mesh currents.
3. Solve the resulting linear system of equations with mesh currents as the independent variables.

In mesh analysis, it is important to be consistent in choosing the direction of current flow. To illustrate the mesh current method, consider the simple two-mesh circuit shown in Figure 2.8. This circuit will be used to generate two equations in the two unknowns, the mesh currents i_1 and i_2. It is instructive to first consider each mesh by itself.

Beginning with mesh 1, note that the voltages around the mesh have been assigned in Figure 2.8 according to the direction of the mesh current, i_1. Recall that as long as signs are assigned consistently, an arbitrary direction may be assumed for any current in a circuit; if the resulting numerical answer for the current is negative, then the chosen reference direction is opposite to the direction of actual current flow. Thus, one need not be concerned about the actual direction of current flow in mesh analysis, once the directions of the mesh currents have been assigned. The correct solution will result, eventually.

According to the sign convention, then, the voltages v_1 and v_2 are defined as shown. Now, it is important to observe that while mesh current i_1 is equal to the current flowing through resistor R_1 (and is therefore also the branch current through R_1), it is not equal to the current through R_2. The branch current through R_2 is the difference between the two mesh currents, $i_1 - i_2$. Thus, since the polarity of the voltage v_2 has already been assigned, according to the convention discussed in the previous paragraph, it follows that the voltage v_2 is given by:

$$v_2 = (i_1 - i_2)R_2 \tag{2.19}$$

Finally, the complete expression for mesh 1 is

$$v_s - i_1R_1 - (i_1 - i_2)R_2 = 0 \tag{2.20}$$

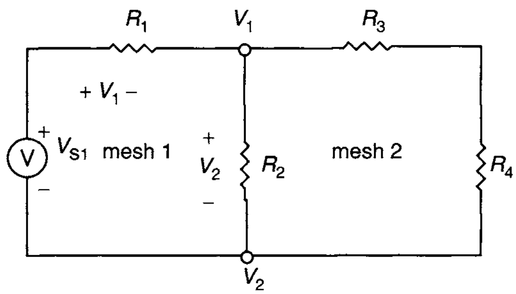

Figure 2.8 Assigning currents and voltages for mesh 1.

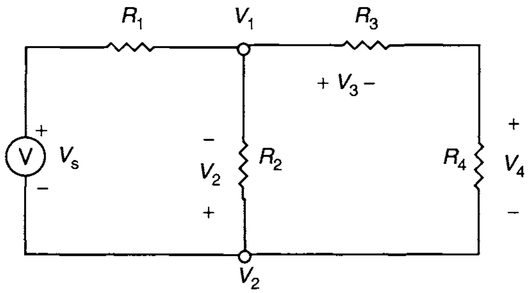

Figure 2.9 Assigning currents and voltages for mesh 2.

The same line of reasoning applies to the second mesh. Figure 2.9 depicts the voltage assignment around the second mesh, following the clockwise direction of mesh current i_2. The mesh current i_2 is also the branch current through resistors R_3 and R_4; however, the current through the resistor that is shared by the two meshes, R_2, is now equal to $(i_2 - i_1)$, and the voltage across this resistor is

$$v_2 = (i_2 - i_1)R_2 \tag{2.21}$$

and the complete expression for mesh 2 is

$$(i_2 - i_1)R_2 + i_2R_3 + i_2R_4 = 0 \tag{2.22}$$

Why is the expression for v_2 obtained in Equation 2.21 different from Equation 2.19? The reason for this apparent discrepancy is that the (clockwise) mesh current dictates the voltage assignment for each mesh. Thus, since the mesh currents flow through R_2 in opposing directions, the voltage assignments for v_2 in the two meshes will also be opposite. This is perhaps a potential source of confusion in applying the mesh current method; you should be very careful to carry out the assignment of the voltages around each mesh separately.

Combining the equations for the two meshes, we obtain the following system of equations:

$$\begin{aligned} (R_1 + R_2)i_1 - i_2R_2 &= v_s \\ -R_2i_1 + (R_2 + R_3 + R_4)i_2 &= 0 \end{aligned} \tag{2.23}$$

These equations may be solved simultaneously to obtain the desired solution, namely, the mesh currents, i_1 and i_2. You should verify that knowledge of the mesh currents permits determination of all the other voltages and currents in the circuit. The following example further illustrates some of the detail of this method.

EXAMPLE 2.4

Mesh current analysis

Figure 2.10 shows a circuit, in which node voltages are:

$$V_{s1} = V_{s2} = 120\,\text{V}$$
$$V_A = 100\,\text{V}$$
$$V_B = -115\,\text{V}$$

Determine the voltage across each resistor.

Solution

Assume a polarity for the voltages across R_1 and R_2 (e.g. from ground to node A, and from node B to ground). R_1 is connected between node A and ground; therefore, the voltage across R_1 is equal to this node voltage. R_2 is connected between node B and ground; therefore, the voltage across R_2 is equal to the negative of this voltage.

$$V_{R1} = V_A = 110\,\text{V}$$
$$V_{R2} = 0 - V_B = 115\,\text{V}$$

The two node voltages are with respect to the ground, which is given. Assume a polarity for the voltage across R_3 (e.g. from node B to node A). Then:

$$\text{By KVL}: \quad V_A + V_{R3} + V_B = 0\,\text{V}$$
$$V_{R3} = V_A + V_B = 110 - (-115) = 225\,\text{V}$$

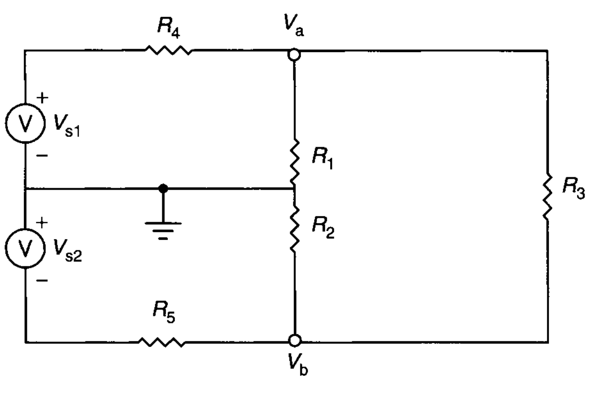

Figure 2.10 A circuit with three meshes.

Assume the polarities for the voltages across R_4 and R_5 (e.g. from node A to ground, and from ground to node B):

$$\text{By KVL}: \quad V_{s1} + V_{R4} + V_A = 0\,\text{V}$$

$$V_{R4} = V_{s1} - V_A = 120 - 110 = 10\,\text{V}$$

$$\text{Also by KVL}: \quad -V_{s2} - V_B - V_{R5} = 0\,\text{V}$$

$$V_{R5} = -V_{s2} - V_B = -120 - (-115) = -5\,\text{V}$$

2.3.3 The principle of superposition

This section briefly discusses a concept that is frequently called upon in the analysis of linear circuits. Rather than a precise analysis technique, such as the mesh current and node voltage methods, the principle of superposition is a conceptual aid that can be very useful in visualizing the behavior of a circuit containing multiple sources. The principle of superposition applies to any linear system and for a linear circuit may be stated as follows:

> In a linear circuit containing $N\psi$ sources, each branch voltage and current is the sum of $N\psi$ voltages and currents, each of which may be computed by setting all but one source equal to zero and solving the circuit containing that single source.

This principle can easily be applied to circuits containing multiple sources and is sometimes an effective solution technique. More often, however, other methods result in a more efficient solution. We consider an example.

EXAMPLE 2.5 Superposition

Figure 2.11 shows a circuit, in which

$$I_B = 10\,\text{A}; \quad V_G = 12\,\text{V};$$
$$R_B = 1.25\,\Omega; \quad R_G = 0.5\,\Omega; \quad R = 0.25\,\Omega$$

Determine the voltage across the resistor R.

Solution

Specify a ground node and the polarity of the voltage across R. Suppress the voltage source by replacing it with a short circuit. Redraw the circuit.

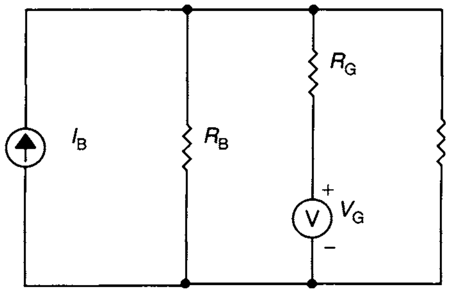

Figure 2.11 A circuit used to illustrate the superposition principle.

By KCL:

$$-I_B + \frac{V_{R-I}}{R_B} + \frac{V_{R-I}}{R_G} + \frac{V_{R-I}}{R} = 0 \tag{2.23A}$$

$$V_{R-I} = \frac{I_B}{\frac{1}{R_B} + \frac{1}{R_G} + \frac{1}{R}} = \frac{10}{\frac{1}{1.25} + \frac{1}{0.5} + \frac{1}{0.25}} = 1.47 \text{ V} \tag{2.23B}$$

Suppress the current source by replacing it with an open circuit.

By KCL:

$$\frac{V_{R-V}}{R_B} + \frac{V_{R-V} - V_G}{R_G} + \frac{V_{R-V}}{R} = 0 \tag{2.23C}$$

$$V_{R-V} = \frac{\frac{V_G}{R_G}}{\frac{1}{R_B} + \frac{1}{R_G} + \frac{1}{R}} = \frac{\frac{12}{0.5}}{\frac{1}{1.25} + \frac{1}{0.5} + \frac{1}{0.25}} = 3.53 \text{ V} \tag{2.23D}$$

$$V_R = V_{R-I} + V_{R-V} = 1.47 + 3.53 = 5 \text{ V} \tag{2.23E}$$

Note: Superposition essentially doubles the work required to solve this problem. The voltage across R can easily be determined using a single KCL.

2.3.4 Thévenin and Norton equivalent circuits

It is always possible to view even a very complicated circuit in terms of much simpler *equivalent* source and load circuits. The analysis of equivalent circuits is more easily managed than the original complex circuit. In studying node voltage and mesh current analysis, you may have observed that there is a certain correspondence (called *duality*) between current sources and voltage sources, on the one hand, and parallel and series circuits, on the other. This duality appears again very clearly in the analysis of equivalent circuits: it will shortly be shown that equivalent circuits fall into one of two classes, involving either a voltage or a current source and, respectively, either series or parallel resistors, reflecting this same principle of duality.

2.3.4.1 *Thévenin's theorem*

As far as a load is concerned, an equivalent circuit consisting of an ideal voltage source, V_T, in series with an equivalent resistance R_T, may represent any network composed of ideal voltage and current sources, and of linear resistors.

2.3.4.2 *Norton's theorem*

As far as a load is concerned, an equivalent circuit consisting of an ideal current source, I_N, in parallel with an equivalent resistance R_N, may represent any network composed of ideal voltage and current sources, and of linear resistors.

2.3.4.3 *Determination of the Norton or Thévenin equivalent resistance*

The first step in computing a Thévenin or Norton equivalent circuit consists of finding the equivalent resistance presented by the circuit at its terminals. This is done by setting all sources in the circuit equal to zero and computing the effective resistance between terminals. The voltage and current sources present in the circuit are set to zero by the same technique used with the principle of superposition: voltage sources are replaced by short circuits, current sources by open circuits. The steps involved in the computation of equivalent resistance are as follows:

1. Remove the load.
2. Set all independent voltage and current sources to zero.
3. Compute the total resistance between load terminals, *with the load removed.* This resistance is equivalent to that which would be encountered by a current source connected to the circuit in place of the load.

To illustrate the procedure, consider the simple circuit of Figure 2.12; the objective is to compute the equivalent resistance the load R_L 'sees' at port *a–b*.

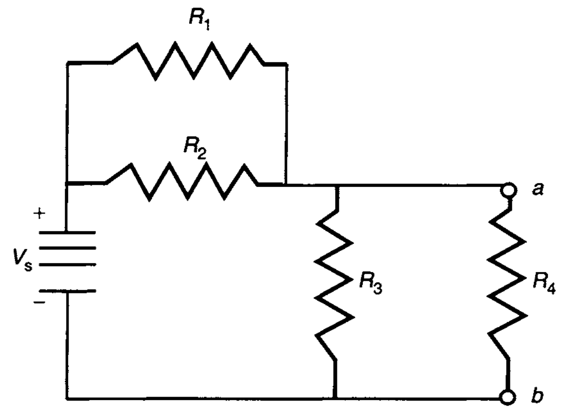

Figure 2.12 Network to illustrate the calculation of the Thévenin resistance.

In order to compute the equivalent resistance, we remove the load resistance from the circuit and replace the voltage source, V_S, by a short circuit. At this point, seen from the load terminals, the circuit appears as shown in Figure 2.13. You can see that R_1 and R_2 are in parallel, since they are connected between the same two nodes. If the total resistance between terminals a and b is denoted by R_T, its value can be determined as follows:

$$R_T = R_1 \| R_2 \| R_3 \tag{2.23F}$$

The equivalent circuit is shown in Figure 2.14, with the source voltage in series with the equivalent Thévenin resistance, so that the voltage seen at a–b is obtained using

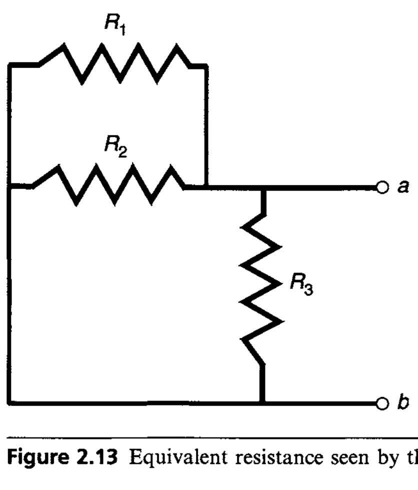

Figure 2.13 Equivalent resistance seen by the load.

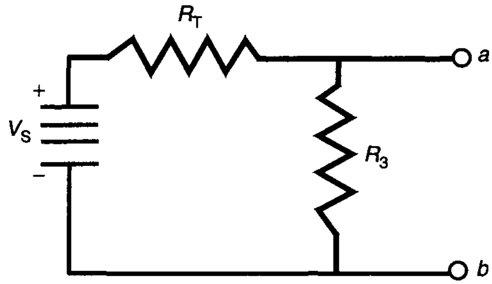

Figure 2.14 Thévenin equivalent circuit of Figure 2.12.

voltage divider equation as

$$V_{TH} = \frac{R_{TH}R_3}{R_{TH} + R_3} V_S \qquad (2.23G)$$

Let us now apply this principle to solve a problem.

EXAMPLE 2.6

Thévenin equivalent circuit

For Figure 2.15 having:

$$V_B = 11\,V; \quad V_G = 12\,V$$
$$R_B = 0.7\,V; \quad R_G = 0.3\,V; \quad R_L = 7.2$$

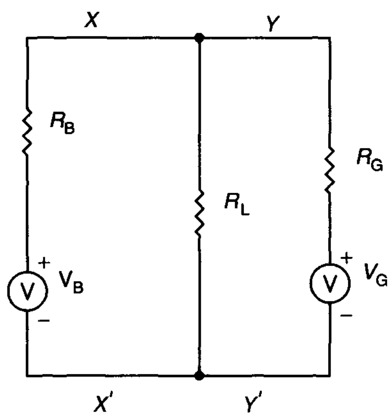

Figure 2.15 Two-mesh, two-source circuit.

Determine:

(a) the Thévenin equivalent of the circuit to the left of $Y\text{–}Y'$;
(b) the voltage between $Y\text{–}Y'$.

Solution

(a) Specify the polarity of the Thévenin equivalent voltage:
Using the voltage divider expression:

$$V_{\mathrm{TH}} = \frac{V_{\mathrm{B}} R_{\mathrm{L}}}{R_{\mathrm{B}} + R_{\mathrm{L}}} = \frac{11 \times 7.2}{0.7 + 7.2} = 10.03\ \mathrm{V} \tag{2.23H}$$

Suppress the generator source:

$$R_{\mathrm{T}} = R_{\mathrm{L}} \| R_{\mathrm{B}} = \frac{7.2 \times 0.7}{0.7 + 7.2} = 638\ \mathrm{m}\Omega \tag{2.23I}$$

(b) Specify the polarity of the terminal voltage. Choose a ground.
Using KCL:

$$\begin{aligned} &\frac{V_{\mathrm{T}} - V_{\mathrm{G}}}{R_{\mathrm{G}}} + \frac{V_{\mathrm{T}} - V_{\mathrm{T}}}{R_{\mathrm{T}}} = 0 \\ &V_{\mathrm{T}} = \frac{\dfrac{V_{\mathrm{G}}}{R_{\mathrm{G}}} + \dfrac{V_{\mathrm{T}}}{R_{\mathrm{T}}}}{\dfrac{1}{R_{\mathrm{G}}} + \dfrac{1}{R_{\mathrm{T}}}} = 11.37\ \mathrm{V} \end{aligned} \tag{2.23J}$$

2.4 Sinusoidal sources and complex impedance

We now consider current and voltage sources with time average values of zero. We will use periodic signals but the observation time could well be less than one period. Periodic signals are also useful in the sense that arbitrary signals can usually be expanded in terms of a Fourier series of periodic signals. Let us start with the following:

$$\begin{aligned} v(t) &= V_{\mathrm{o}} \cos\ t(\omega t + \phi_{\mathrm{V}}) \\ i(t) &= I_{\mathrm{o}} \cos\ t(\omega t + \phi_{\mathrm{I}}) \end{aligned} \tag{2.24}$$

Notice that we have now switched to lowercase symbols. Lowercase is generally used for a.c. quantities while uppercase is reserved for d.c. values. Now is the time to get into complex notation, often used in electrical and electronic equations, since it will make our discussion easier. The above voltage and current signals can be written as

$$\vec{v}(t) = V_0 e^{j(\omega t + \phi_V)}$$
$$\vec{i}(t) = I_0 e^{j(\omega t + \phi_I)} \tag{2.25}$$

In order to make things easier, we define one EMF in the circuit to have $\varphi = 0$. In other words, we will pick $t = 0$ to be at the peak of one signal. The vector notation is used to remind us that complex numbers can be considered as vectors in the complex plane. Although not so common in physics, in electronics we refer to these vectors as phasors. Hence the reader should now review complex notation. The presence of sinusoidal $\vec{v}(t)$ or $\vec{i}(t)$ in circuits will result in an inhomogeneous differential equation with a time-dependent source term. The solution will contain sinusoidal terms with the source frequency. The extension of Ohm's law to a.c. circuits can be written as

$$\vec{v}(\omega, t) = Z(\omega)\vec{i}(\omega, t), \tag{2.26}$$

where ω is the source frequency. The generalized resistance referred to as the impedance is represented by the letter Z. We can cancel out the common time-dependent factors to obtain

$$\vec{v}(\omega) = Z(\omega)\vec{i}(\omega) \tag{2.27}$$

and hence the power of the complex notation becomes obvious. For a physical quantity we take the amplitude of the real signal as follows

$$\left|\vec{v}(\omega)\right| = \left|Z(\omega)\right|\left|\vec{i}(\omega)\right| \tag{2.28}$$

We will now examine each circuit element in turn with a voltage source to deduce its impedance.

2.4.1 Resistive impedance

For a voltage source and resistor, the impedance is equal to the resistance, as expected, given as

$$Z = R \tag{2.29}$$

2.4.2 Capacitive impedance

For a voltage source and capacitor, the impedance is given as

$$Z = \frac{1}{j\omega C}. \tag{2.30}$$

For d.c. circuits $\omega = 0$ and hence $Z_c \to \infty$. The capacitor acts like an open circuit (infinite resistance) in a d.c. circuit.

2.4.3 Inductive impedance

For a voltage source and an inductor, the impedance is given as

$$Z = j\omega L \tag{2.31}$$

For d.c. circuits $\omega = 0$ and hence $Z_L \to \infty$. There is no voltage drop across an inductor in a d.c. circuit.

EXAMPLE 2.7 Alternating current circuit

For Figure 2.16 having:

$$R_1 = 100\,\Omega; \quad R_2 = 50\,\Omega; \quad L = 10\,\text{mH}; \quad C = 10\,\mu\text{F};$$

$$v_s(t) = 5\ \cos\left(10\,000t + \frac{\pi}{2}\right) \tag{2.31A}$$

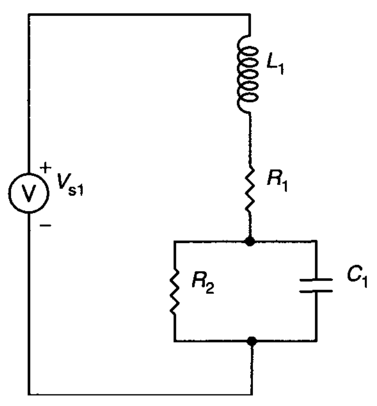

Figure 2.16 Alternating current example.

Determine:

(a) the equivalent impedance of the circuit;
(b) the source current.

Solution

Considering R_2 and C:

$$Z_C = \frac{1}{j\omega C} \qquad (2.31B)$$

$$Z_{\|} = \frac{Z_R Z_C}{Z_R + Z_C} = \frac{R\frac{1}{j\omega C}}{R + j\omega C} = \frac{R}{1 + j\omega RC} \qquad (2.31C)$$

$$Z_{\|} = \frac{50}{1 + j \times 10^4 \times 50 \times 10 \times 10^{-6}}$$

$$= \frac{50}{1 + j5} = \frac{50(1 - j5)}{(1 + j5)(1 - j5)} = 1.92 - j9.62$$

$$r = \sqrt{(1.92^2 + 9.62^2)} = 9.81;$$

$$\theta = \tan^{-1}\left(\frac{-9.62}{1.92}\right) = -78.7^\circ = \frac{-78.7\pi}{180} = -1.3734^c \qquad (2.31D)$$

$$Z_{\|} = 9.81\angle - 1.3734$$

$$Z = Z_{R1} + Z_L + Z_{\|}$$

$$= 100 + j10^4 \times 10 \times 10^{-3} + 1.92 - 9.6j = 101.92 + 90.38j$$

$$r = \sqrt{(101.92^2 + 90.38^2)} = 136.22;$$

$$\theta = \tan^{-1}\left(\frac{90.38}{101.92}\right) = 41.56^\circ = \frac{41.56\pi}{180} = 0.7254^c \qquad (2.31E)$$

$$Z = 136.221\angle 0.7254$$

The source current can now be computed as:

$$I = \frac{V_S}{Z} = \frac{5\angle\pi/2}{136.221\angle 0.7254} = \frac{5\angle 1.578}{136.221\angle 0.7254} = 36\angle 0.8453 \text{ mA} \tag{2.31F}$$
$$i(t) = 36 \cos(10\,000t + 0.8453)$$

EXAMPLE 2.8

See website for downloadable MATLAB code to solve this problem

Passive element circuit

Two fuses F_1 and F_2 (Figure 2.17), under normal conditions, are modeled as short circuits. However, if excess current flows through a fuse, it melts and consequently blows (becoming an open circuit).

Determine, using KVL, and mesh analysis, the voltages across R_1, R_2 and R_3 under normal condition (no blown fuses), when

$$V_{s1} = 115\,\text{V}; \quad V_{s2} = 115\,\text{V};$$
$$R_1 = R_2 = 5\,\Omega; \quad R_3 = 10\,\Omega; \quad R_4 = R_5 = 200\,\text{m}\Omega$$

Solution

Using KVL:

$$I_1(R_1 + R_4) - I_3R_1 = V_{s1}$$
$$I_2(R_2 + R_5) - I_3R_2 = V_{s2}$$
$$-I_1R_1 - I_2R_2 + I_3(R_1 + R_2 + R_3) = 0$$

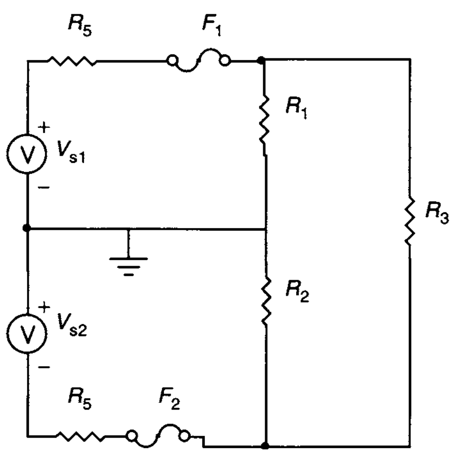

Figure 2.17 Fused circuit.

Substituting resistor values and rearranging:

$$\begin{array}{lll} 5.2I_1 & 0 & -5I_3 = 115 \\ 0 & 5.2I_2 & -5I_3 = 115 \\ 5I_1 & +5I_2 & -20I_3 = 0 \end{array}$$

$$I_1 = I_2 \qquad I_1 = 2I_3$$

$$\Delta = \begin{vmatrix} 5.2 & 0 & -5 \\ 0 & 5.2 & -5 \\ 5 & 5 & -20 \end{vmatrix} = 280.8$$

$$I_1 = I_2 = \frac{\begin{vmatrix} 115 & 0 & -5 \\ 115 & 5.2 & -5 \\ 0 & 5 & -20 \end{vmatrix}}{\Delta} = \frac{-11960}{-280.8} = 42.6\,\text{A} \qquad (2.31\text{G})$$

$$I_3 = I_1/2 = 21.3\,\text{A}$$

$$V_{R1} = R_1(I_1 - I_3) = 5 \times 21.3 = 106.5\,\text{V}$$

$$V_{R2} = R_2(I_3 - I_2) = -5 \times 21.3 = -106.5\,\text{V}$$

$$V_{R3} = I_3R_3 = 10 \times 21.3 = 213\,\text{V}$$

EXAMPLE 2.9

See website for downloadable MATLAB code to solve this problem

Passive element circuit

Determine, using KVL, and mesh analysis, the voltages across R_1, R_2 and R_3 in Figure 2.17 (Example 2.8) for the following parameter values:

$$V_{s1} = 110\,\text{V}; \quad V_{s2} = 110\,\text{V};$$
$$R_1 = 100\,\Omega; \quad R_2 = 22\,\Omega; \quad R_3 = 70\,\Omega; \quad R_4 = R_5 = 13\,\Omega.$$

Solution

Using KVL:

$$\begin{aligned} I_1(R_1 + R_4) - I_3R_1 &= V_{s1} \\ I_2(R_2 + R_5) - I_3R_2 &= V_{s2} \\ -I_1R_1 - I_2R_2 + I_3(R_1 + R_2 + R_3) &= 0 \end{aligned} \qquad (2.31\text{H})$$

Substituting resistor values and rearranging:

$$\begin{array}{lll} 113I_1 & 0 & -100I_3 = 110 \\ 0 & 35I_2 & -22I_3 = 110 \\ 100I_1 + & 22I_2 & -192I_3 = 0 \end{array}$$

$$\Delta = \begin{vmatrix} 113 & 0 & -100 \\ 0 & 35 & -22 \\ 100 & 22 & -192 \end{vmatrix} = -354\,668$$

$$I_1 = \frac{\begin{vmatrix} 110 & 0 & -100 \\ 110 & 35 & -22 \\ 0 & 22 & -192 \end{vmatrix}}{\Delta} = \frac{-935\,660}{-354\,668} = 2.64\,\text{A} \tag{2.31 I}$$

$$I_2 = 4.31\,\text{A}$$

$$I_3 = 1.86\,\text{A}$$

$$V_{R1} = R_1(I_1 - I_3) = 100 \times 0.76 = 76\,\text{V}$$

$$V_{R2} = R_2(I_3 - I_2) = -22 \times 2.43 = -53.46\,\text{V}$$

$$V_{R3} = I_3 R_3 = 70 \times 1.88 = 131.6\,\text{V}$$

Problems

Node and mesh analysis methods

Q2.1 Determine the voltage across R_5 in Figure 2.18 when

$V_{s1} = 4\,\text{V}$; $V_{s2} = 2\,\text{V}$; $R_1 = 2\,\text{k}\Omega$; $R_2 = 4\,\text{k}\Omega$; $R_3 = 4\,\text{k}\Omega$; $R_4 = 2\,\text{k}\Omega$; $R_5 = 6\,\text{k}\Omega$; $R_6 = 2\,\text{k}\Omega$.

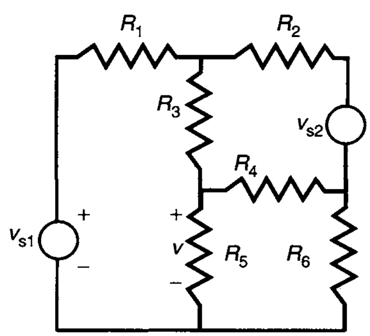

Figure 2.18 Circuit for Q2.1.

Q2.2 For the circuit in Figure 2.19, V_{S2} and R_S model a temperature sensor, and the voltage R_3 indicates the temperature. Determine the temperature.

$$V_{s1} = 24\,\text{V}; \quad V_{s2} = \text{kT}, \quad \text{where } k = 15\,\text{V/°C}; \quad V_{R3} = -4\,\text{V}$$
$$R_1 = R_s = 15\,\text{k}\Omega; \quad R_2 = 5\,\text{k}\Omega; \quad R_3 = 10\,\text{k}\Omega; \quad R_4 = 24\,\text{k}\Omega.$$

Q2.3 For the circuit in Figure 2.20 having $V_S = 10\,\text{V}$, $A_V = 50$, $R_1 = 3\,\text{k}\Omega$; $R_2 = 8\,\text{k}\Omega$; $R_3 = 2\,\text{k}\Omega$; $R_4 = 0.3\,\text{k}\Omega$, determine the voltage across R_4 using KCL and node analysis.

Q2.4 Determine (a) the current I, and (b) the voltage at node A, in Figure 2.21, where

$$V_1 = 5\,\text{V}; \quad V_2 = 10\,\text{V}$$
$$R_1 = 1\,\text{k}\Omega; \quad R_2 = 8\,\text{k}\Omega; \quad R_3 = 10\,\text{k}\Omega;$$
$$R_4 = 2\,\text{k}\Omega; \quad R_5 = 2\,\text{k}\Omega$$

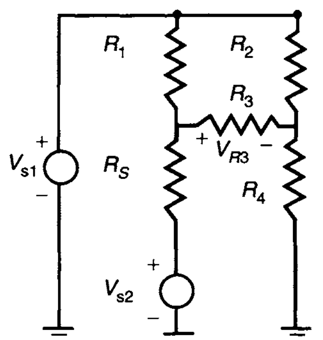

Figure 2.19 Circuit for Q2.2.

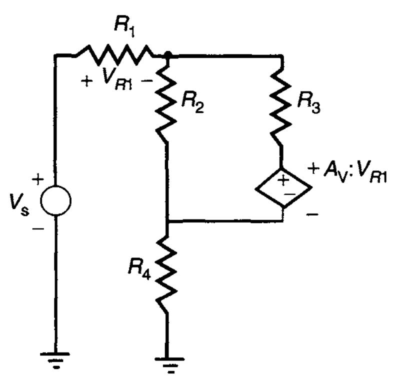

Figure 2.20 Circuit for Q2.3.

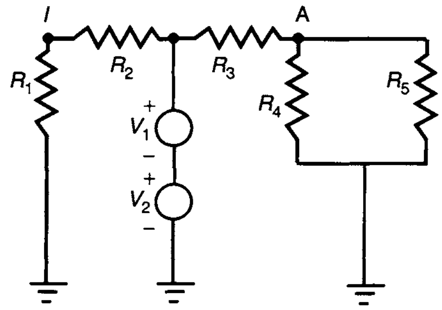

Figure 2.21 Circuit for Q2.4.

Q2.5 In Figure 2.22, F_1 and F_2 are fuses. Under normal conditions they are modeled as short circuits. However, if excess current flows through a fuse, it melts and consequently blows (becoming an open circuit). The component values are:

$V_{s1} = 110\,V$; $V_{s2} = 110\,V$;
$R_1 = 100\,\Omega$; $R_2 = 25\,\Omega$; $R_3 = 75\,\Omega$; $R_4 = R_5 = 15\,\Omega$.

Normally, the voltages across R_1, R_2, and R_3 are 106.5 V, −106.5 V, and 213 V, respectively. If fuse F_1 now blows, or opens, determine, using KVL, and mesh analysis, the new voltages across R_1, R_2 and R_3.

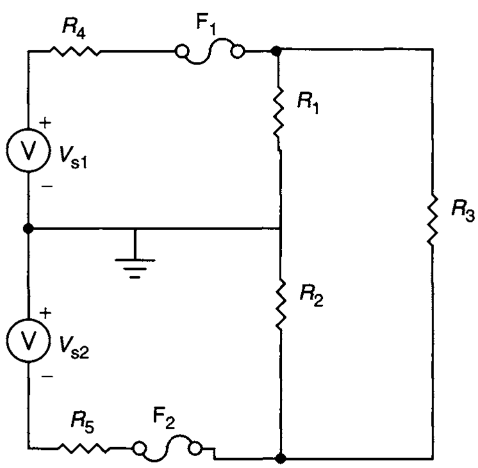

Figure 2.22 Circuit for Q2.5.

Thévenin and Norton equivalent circuits

Q2.6 In Figure 2.23, $V_S = 12\,V$; $R_1 = 7\,k\Omega$; $R_2 = 3\,k\Omega$; $R_3 = 8\,k\Omega$; $R_4 = 6\,k\Omega$. Determine:

(a) the Thévenin equivalent of the circuit to the left of *a–b*;

(b) the voltage across *a–b*.

Q2.7 In the circuit shown in the Figure 2.24:

$$V_1 = 15\,V;\quad V_2 = 12\,V;\quad I = 20\,mA;$$
$$R_1 = 10\,k\Omega;\quad R_2 = 2\,k\Omega;\quad R_3 = 4\,k\Omega;\quad R_4 = 5\,k\Omega;$$
$$R_5 = 8\,k\Omega;\quad R_6 = 6\,k\Omega;\quad R_7 = 3\,k\Omega;$$
$$C = 20\,\mu F.$$

Determine the Norton equivalent circuit with respect to *C*.

Sinusoidal sources

Q2.8 For the circuit shown in Figure 2.25, determine, for the values given,

(a) the equivalent impedance of the circuit the source current;

(b) the source current.

$$R_1 = 200\,\Omega;\quad R_2 = 100\,\Omega;\quad L = 50\,mH;\quad C = 50\,\mu F;$$
$$v_s(t) = 5\ \cos\left(5000t + \frac{\pi}{4}\right). \tag{2.31J}$$

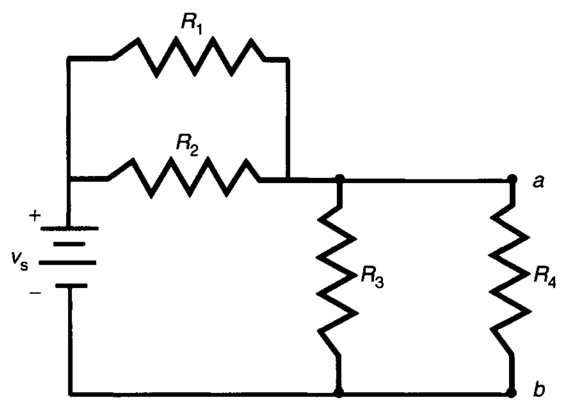

Figure 2.23 Circuit for Q2.6.

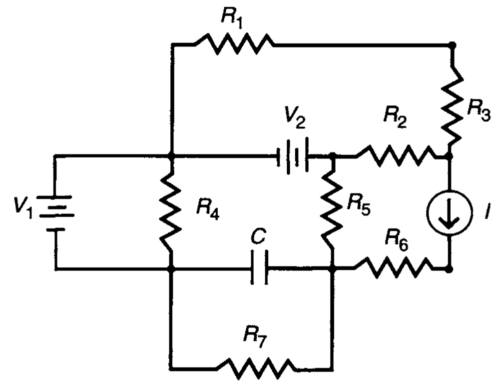

Figure 2.24 Circuit for Q2.7.

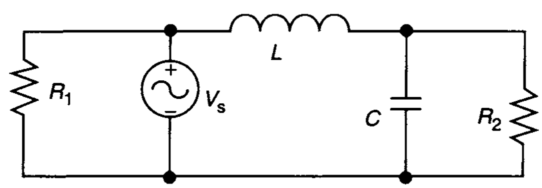

Figure 2.25 Circuit for Q2.8.

Further reading

[1] Horowitz, P. and Hill, W. (1989) *The Art of Electronics* (2nd. ed.), New York: Cambridge University Press.
[2] Rashid, M.H. (1996) *Power Electronics: Circuits, Devices, and Applications*, Prentice Hall.
[3] Rizzoni, G. (2003) *Principles and Applications of Electrical Engineering* (4th. ed.), McGraw-Hill.